Lecture 10

Properties of Expectation

1. For constants $a, b \in \mathbb{R}$,
$$E[aX + b] = a\,E[X] + b.$$

2. For constants $a, b \in \mathbb{R}$,
$$\mathrm{Var}(aX + b) = E[(aX + b)^2] - \big(E[aX + b]\big)^2 = a^2\,\mathrm{Var}(X).$$

Moments

- The **first moment** is the expectation:
$$E[X].$$

- The **second central moment** is the variance:
$$\mathrm{Var}(X).$$

- In general, the r-th central moment about the mean is
$$\mu_r = E\big[(X - E[X])^r\big].$$

Existence of Moments

If $E[X^r]$ exists for some r, then $E[X^s]$ also exists for all $s \leq r$.

Moment Generating Function (MGF)

Let X be a continuous random variable with pdf $f_X(x)$. The Moment Generating Function (MGF) of X, denoted by $M_X(t)$, is defined as:

$$M_X(t) = E\left(e^{tX}\right),$$

provided the expectation exists.

Example 1

$$X : -1,\ 0,\ 1$$
$$P_X : \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{2}$$

Mean:

$$E(X) = \frac{1}{4}$$

Second Moment:

$$E(X^2) = \frac{3}{4}$$

Variance:

$$\mathrm{Var}(X) = \frac{3}{4} - \left(\frac{1}{4}\right)^2 = \frac{11}{16}$$

Moment Generating Function

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx}\, P_X$$

$$M_X(t) = e^{-t} \cdot \tfrac{1}{4} \ +\ e^{0\cdot t} \cdot \tfrac{1}{4} \ +\ e^{1\cdot t} \cdot \tfrac{1}{2}$$

$$M_X(t) = \frac{1}{4}e^{-t} + \frac{1}{4} + \frac{1}{2}e^{t}$$

The pdf is

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

The MGF is

$$M_X(t) = \int_0^2 e^{tx} \cdot \frac{1}{2}\, dx.$$

$$M_X(t) = \frac{1}{2} \cdot \left[\frac{e^{tx}}{t}\right]_0^2 = \frac{1}{2}\left(\frac{e^{2t}-1}{t}\right), \quad t \neq 0.$$

Thus,

$$M_X(t) = \frac{e^{2t}-1}{2t}, \quad t \neq 0.$$

Example 2

The pdf is

$$f(x) = e^{-x}, \quad x > 0.$$

The MGF is

$$M_X(t) = \int_0^{\infty} e^{-x} e^{tx}\, dx = \int_0^{\infty} e^{-(1-t)x}\, dx.$$

This integral converges only when $t < 1$.

$$M_X(t) = \left[\frac{-1}{1-t} e^{-(1-t)x} \right]_0^{\infty} = \frac{1}{1-t}, \quad t < 1.$$

So,

$$M_X(t) = \frac{1}{1-t}, \quad t < 1.$$

For $t < 1$, the MGF of X exists and we can expand:

$$M_X(t) = E[e^{tX}] = E\left[1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \cdots \right].$$

$$M_X(t) = 1 + t\, E[X] + \frac{t^2}{2!}\, E[X^2] + \cdots$$

From the series expansion,

$$E[X] = \frac{d}{dt} M_X(t) \bigg|_{t=0},$$

$$E[X^n] = \frac{d^n}{dt^n} M_X(t) \bigg|_{t=0}.$$

Result

If X and Y are two random variables such that

$$M_X(t) = M_Y(t),$$

then X and Y are **identically distributed**, that is, both X and Y follow same probability distribution.

Example

Let

$$X : -1,\ 0,\ 1, \quad P(X) = \tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}.$$

Then,

$$E[X] = (-1)\cdot\tfrac{1}{3} + 0\cdot\tfrac{1}{3} + (1)\cdot\tfrac{1}{3} = 0.$$

The moment generating function (MGF) may **not exist for all values of** t. For some distributions, the integral

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

diverges for certain t.

Thus,

$$|M_X(t)| \nleq \infty \quad \text{(MGF need not be always bounded).}$$

Characteristic Function (CF)

The **characteristic function** of a random variable X is defined as

$$\varphi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx.$$

Property

Since $|e^{itx}| = 1$,

$$|\varphi_X(t)| \le \int_{-\infty}^{\infty} |e^{itx}| f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Hence,

$$|\varphi_X(t)| \le 1 \quad \text{for all } t.$$

Note:

- MGFs may not exist always.
- Characteristic functions always exist.