Lecture 12

Standard Discrete Distributions

1. Bernoulli Distribution (Binary outcomes)

- Random variable:
$$X \in \{0, 1\}$$
- Probability mass function (pmf):
$$P(X = 1) = p, \quad P(X = 0) = 1 - p$$

x	0	1
$P(X = x)$	$1 - p$	p

General form:
$$P(X = r) = p^r(1 - p)^{1-r}, \quad r \in \{0, 1\}$$
- Expectation:
$$E(X) = p$$
- Variance:
$$\mathrm{Var}(X) = p(1 - p)$$
- Moment generating function (MGF):
$$M_X(t) = E\left(e^{tX}\right) = (1 - p) + pe^t$$

Remark: A *Bernoulli trial* is a trial where there are exactly two possible outcomes (success/failure).

Binomial Distribution

Consider a random variable X = number of successes out of n Bernoulli trials. Here, n is the total number of trials in an experiment.

The experiment is set as:

1. All the trials are independent.
2. All the trials are identical, i.e., the probability of success in each trial is the same, denoted by p.

Let $X_i \sim \text{Bernoulli}(p)$, where

$$P(X_i = 1) = p, \quad P(X_i = 0) = 1 - p.$$

Then,

$$X = X_1 + X_2 + \cdots + X_n$$

is the sum of n Bernoulli trials, called a *Binomial random variable*.

$$X \in \{0, 1, 2, \ldots, n\}$$

The pmf of X is given by:

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}, \quad r = 0, 1, 2, \ldots, n$$

$$\begin{aligned} P(X = 0) &= (1 - p)^n \\ P(X = 1) &= \binom{n}{1} p (1 - p)^{n-1} \end{aligned}$$

- A random variable X is said to follow a **Binomial distribution** with parameters n and p, denoted as
$$X \sim \text{Binomial}(n, p).$$

Example

Suppose India vs. Australia, 5 matches. Let

$$X \sim \text{Binomial}(5, p).$$

Then,

$$P(X = r) = \binom{5}{r} p^r (1 - p)^{5-r}, \quad r = 0, 1, 2, \ldots, 5.$$

- $P(X \leq 2) \neq P(X < 2)$
- $P(X \leq 3) \neq P(X < 3) = P(X \leq 2)$

$$E(X) = \sum_{r=0}^{n} r \binom{n}{r} p^r (1 - p)^{n-r}.$$

Simplifying, we get

$$E(X) = np.$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = np(1-p).$$

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{n} e^{tr} \binom{n}{r} p^r (1-p)^{n-r}.$$

Simplifying,

$$M_X(t) = \left(pe^t + (1-p)\right)^n.$$

Poisson Distribution as a Limit of Binomial

Consider

$$X_n \sim \mathrm{Binomial}(n, p).$$

Conditions

- n is large ($n \to \infty$),
- p is small ($p \to 0$),
- such that $\lambda = np$ remains finite.

Binomial pmf

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}.$$

Substitute $p = \frac{\lambda}{n}$:

$$P(X = r) = \frac{n(n-1)\cdots(n-r+1)}{r!} \left(\frac{\lambda}{n}\right)^r \left(1 - \frac{\lambda}{n}\right)^{n-r}.$$

Limit

As $n \to \infty$,

$$\frac{n(n-1)\cdots(n-r+1)}{n^r} \to 1,$$

and

$$\left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}.$$

Thus,

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}, \quad r = 0, 1, 2, \ldots$$