# Lecture 14

# Geometric Distribution

**Definition 8** (Number of failures before the first success). *Let the random variable $X$ be the number of failures before the first success occurs. The probability mass function (PMF) is given by:*

$$P(X = r) = (1 - p)^r p \quad \text{for } r = 0, 1, 2, \ldots$$

**Definition 9** (Number of trials to get the first success). *Let the random variable $X$ be the number of trials required to get the first success. The PMF is given by:*

$$P(X = r) = (1 - p)^{r-1} p \quad \text{for } r = 1, 2, 3, \ldots$$

**Note 8.1.** *We will use the second definition (number of trials) for the following examples and properties. Let $q = 1 - p$.*

**Example 11** (UPSC Qualification). *Let the probability of qualifying for the UPSC exam in a single attempt be $p = 0.3$. Let $X$ be the number of attempts needed to qualify. What is the probability that exactly 4 attempts are needed?*

*Solution.* Here, $r = 4$, $p = 0.3$, and $q = 0.7$.

$$P(X = 4) = q^{4-1} p = (0.7)^3 (0.3) = 0.343 \times 0.3 = 0.1029$$

□

### 8.1.1 Key Properties and Formulas

For $X \sim \text{Geometric}(p)$ based on the number of trials:

- **Expected Value:** $E[X] = \sum_{r=1}^{\infty} rpq^{r-1} = \frac{1}{p}$
- **Variance:** $\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{q}{p^2}$
- **Moment Generating Function (MGF):**

$$\begin{aligned} M_X(t) = E[e^{tX}] &= \sum_{r=1}^{\infty} e^{tr} pq^{r-1} \\ &= pe^t \sum_{r=1}^{\infty} (qe^t)^{r-1} \\ &= \frac{pe^t}{1 - qe^t}, \quad \text{provided } qe^t < 1 \end{aligned}$$

**Property 8.1** (Memoryless Property). *The Geometric distribution is the only discrete distribution that has the memoryless property.*

$$P(X > m + n \mid X > m) = P(X > n)$$

*This means that, given the first $m$ trials were failures, the probability of having at least $n$ more failures is the same as the original probability of having at least $n$ failures.*

*Proof of the Memoryless Property.* The cumulative distribution function (CDF) is $F_X(r) = P(X \leq r) = 1 - q^r$.
Therefore, the survival function is $P(X > r) = 1 - P(X \leq r) = q^r$.

By the definition of conditional probability:

$$
\begin{aligned}
P(X > m + n \mid X > m) &= \frac{P((X > m + n) \cap (X > m))}{P(X > m)} \\
&= \frac{P(X > m + n)}{P(X > m)} \quad \text{(since if } X > m + n, \text{ it must be that } X > m) \\
&= \frac{q^{m+n}}{q^m} \\
&= q^n \\
&= P(X > n)
\end{aligned}
$$

□

## 8.2 The Negative Binomial (Pascal) Distribution

**Definition 10.** *Let the random variable $X$ be the number of trials required to achieve a total of $k$ successes. The PMF is given by:*

$$P(X = r) = \binom{r-1}{k-1} p^k (1-p)^{r-k} \quad \textit{for } r = k, k+1, k+2, \ldots$$

*The logic is that the $r^{th}$ trial must be the $k^{th}$ success, and the preceding $r - 1$ trials must contain exactly $k - 1$ successes.*

**Note 8.2.** *The Negative Binomial distribution does not satisfy the memoryless property.*

# Standard Continuous Distributions

## The Uniform Distribution

A random variable $X$ follows a Uniform distribution on the interval $[a, b]$, denoted $X \sim U[a, b]$, if it is equally likely to take any value in this interval.

- **Probability Density Function (PDF):**

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- **Cumulative Distribution Function (CDF):**

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

- **Expected Value:** $E[X] = \frac{a+b}{2}$
- **Variance:** $\text{Var}(X) = \frac{(b-a)^2}{12}$

## The Exponential Distribution

The Exponential distribution is the continuous analogue of the Geometric distribution and is often used to model the time until an event occurs.

- **Probability Density Function (PDF):** Let the rate parameter be $\lambda > 0$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Sometimes, it is parameterized with the scale parameter $\beta = 1/\lambda$:

$$f_X(x) = \frac{1}{\beta} e^{-x/\beta} \quad \text{for } x \ge 0$$