Lecture 15

Gamma Function

The Gamma function, $\Gamma(n)$, is useful for integration.

- $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$
- For a positive integer n, $\Gamma(n) = (n-1)!$
- A useful property is $\Gamma(n) = (n-1)\Gamma(n-1)$
- A general form: $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$

Gamma Distribution

Let $X \sim G(\alpha, \beta)$. The PDF is:

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0, \alpha > 0, \beta > 0$$

The Gamma distribution is a family of distributions.

Special Cases

- When $\alpha = 1$, the Gamma distribution becomes the Exponential distribution:

$$f_X(x) = \frac{1}{\Gamma(1)\beta^1} x^{1-1} e^{-x/\beta} = \frac{1}{\beta} e^{-x/\beta}$$

- When $\alpha = 2$:

$$f_X(x) = \frac{1}{\Gamma(2)\beta^2} x^{2-1} e^{-x/\beta} = \frac{x}{\beta^2} e^{-x/\beta}$$

Mean: $E(X) = \alpha\beta$

Moments and MGF

The h^{th} moment for the exponential distribution is:

$$E(X^h) = \frac{1}{\lambda} \int_0^\infty x^h e^{-x/\lambda} dx$$

For the Gamma distribution:

- **Mean:** $E(X) = \alpha\beta$
- **Second Moment:** $E(X^2) = \alpha(\alpha+1)\beta^2$
- **Variance:** $Var(X) = E(X^2) - [E(X)]^2 = \alpha\beta^2$
- **MGF:** $M_X(t) = \frac{1}{(1-\beta t)^\alpha}$, for $t < \frac{1}{\beta}$

Chi-Square (χ^2) Distribution

The Chi-square distribution is a special case of the Gamma distribution where $\alpha = \nu/2$ and $\beta = 2$.

$$f_X(x) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu/2-1} e^{-x/2}, \quad x \geq 0$$

Here, ν is the **degree of freedom**.

- **Mean:** $E(X) = \nu$
- **Variance:** $Var(X) = 2\nu$