Lecture 16

Gaussian (Normal) Distribution

A continuous random variable X follows a normal distribution with a mean μ and variance σ^2 If its probability density function (pdf) is.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

For a normal distribution:

- **Mean (E(X))** $= \mu$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\, dx$$

  - Put $\frac{X-\mu}{\sigma} = Z$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) \exp\left(-\frac{z^2}{2}\right)$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right)\, dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \exp\left(-\frac{z^2}{2}\right)\, dz$$

  - Where $\int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right)\, dz = \sqrt{2\pi}$
  - $\int_{-\infty}^{\infty} z \exp\left(-\frac{Z^2}{2}\right)\, dz = 0$

$$\implies E(X) = \mu$$

- **Variance** $= \sigma^2$
- **MGF** $= E(e^{Xt}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Standardization of a random Variable

Let X be a random variable with mean $\mu = \mathbb{E}[X]$ and variance $\sigma^2 = \mathrm{Var}(X) > 0$, The **standardization** transforms X into a new variable Z and is defined as:

$$Z = \frac{X-\mu}{\sigma}$$

and follows a standard normal distribution:

$$Z \sim \mathcal{N}(0, 1).$$

Properties

- Z has mean 0:
$$E[Z] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\left(E[X]-\mu\right) = 0.$$
- Z has variance 1:
$$Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(X) = 1.$$
- Z has Coefficient of Skewness 0:
$$E[Z^3] = 0$$
- Z has Coefficient of Kurtosis 0:
$$E[Z^4] = 3$$