Lecture 2

2.8 Probability Function

Definition 4. *A **probability function** is a mapping:*

$$P : \mathcal{F} \to [0, 1]$$

used to quantify uncertainty in experiments.

Example 3. *Analyzing data of students in IITJ:*

- *Heights*
- *Weights*
- *Health center visits*
- *Academic perceptions*

Each variable exhibits different levels of uncertainty. For example, in height analysis:

$$\textit{Minimum height} = 3 \textit{ ft}, \quad \textit{Maximum height} = 7 \textit{ ft}$$

We can use bar charts or histograms to understand the uncertainty.

2.9 Axioms of Probability

Let S be the **sample space**. Then $P : F \to [0, 1]$ be such that

1. $P(S) = 1$
2. If $A_1, A_2, \cdots \subset S$ are such that $A_i \cap A_j = \varnothing$ for $i \neq j$, then:

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

2.10 Properties of the Power Set

For the **power set** $\mathcal{P}(S)$:

1. $\varnothing \in \mathcal{P}(S)$
2. $S \in \mathcal{P}(S)$
3. If $A \in \mathcal{P}(S)$, then $A^c \in \mathcal{P}(S)$
4. If $A, B \in \mathcal{P}(S)$, then $A \cup B \in \mathcal{P}(S)$ and $A \cap B \in \mathcal{P}(S)$

2.11 Sigma Algebra

Definition 5. *A* ***sigma algebra*** $\mathcal{F}$ *is a set of subsets of* S *such that:*

1. $\varnothing \in \mathcal{F}$
2. *If* $A \in \mathcal{F}$*, then* $A^c \in \mathcal{F}$
3. *If* $A, B \in \mathcal{F}$*, then* $A \cup B \in \mathcal{F}$

Remark 2 (De Morgan's Law)**.** *Union and complement closure imply intersection closure:*

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{F}$$

Proposition 1. *The power set* $\mathcal{P}(S)$ *is maximal sigma algebra.*

Example 4. $S = \{1, 2, \ldots, 6\}$*, throwing a die. Let* $A = \{1, 3, 5\}$*. Define* $\mathcal{F} = \{\varnothing, S, A, A^c\}$*.*

Example 5. $S = \{1, 2, 3, 4, 5, 6\}$*,* $A = \{2, 4, 6\}$*,* $B = \{3, 6\}$*. Find the smallest sigma algebra* $\mathcal{F}$ *containing* A *and* B*.*

Example 6. $\mathcal{F} = \{\varnothing, S\}$ *is the minimal sigma algebra.*

2.12 Properties of Probability Functions

1. $P(A^c) = 1 - P(A)$
2. $P(S) = 1$
3. $P(A \cup A^c) = 1 \implies P(\varnothing) = 0$ if $A = S$

2.13 Set Operations in Probability

Using union, intersection, and complement:

$$A \cup B : \text{“A or B”}, \quad A \cap B : \text{“A and B”}, \quad A^c : \text{“not A”}$$

If $A \cap B = \varnothing$, then A and B are **mutually exclusive**.

If $A \cap B \neq \varnothing$, then they are **not mutually exclusive** and:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof. Write $A = (A \cap B) \cup (A \cap B^c)$ and $B = (B \cap A) \cup (B \cap A^c)$. Adding:

$$P(A) + P(B) = P(A \cap B) + P(A \cap B^c) + P(B \cap A) + P(B \cap A^c)$$

Simplifying gives the required formula. □

Theorem 1. *For three events:*

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

2.14 Independence and Conditional Probability

If A and B are independent:

$$P(A \cap B) = P(A)P(B)$$

If dependent:

$$P(A \cap B) = P(A)P(B|A)$$

where $P(B|A)$ is the **conditional probability** of B given A.

- The process of conditioning reduces the sample space. For example, searching for a student in IITJ given that the student is in the Lecture Hall Complex (LHC).