Lecture 7

Example

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ kx^3, & 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution: Using the definition of PDF, we have

$$\int_{-\infty}^{\infty} f(x)dx = 1$$
$$\int_0^1 xdx + \int_1^2 kx^3 dx = 1 \Longrightarrow k = \frac{2}{15}$$

Now CDF is

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{2}, & 0 \leq x \leq 1, \\ \frac{1}{2} + \frac{1}{30}(x^4 - 1), & 1 \leq x \leq 2, \\ 1, & \text{otherwise.} \end{cases}$$

Functions of Random Variables

Let X has probability distribution

$$\begin{array}{lllll} X: 0 & 1 & 2 & 3 \\ P_X: \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

What is the probability distribution of $Y = g(X)$?.
Let $Y = X^2$

$$\begin{array}{lllll} Y: 0 & 1 & 4 & 9 \\ P_Y: \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

Consider the following distribution

$$\begin{array}{lll} X: -1 & 1 & 1 \\ P_X: \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}$$

Let $Y = X^2$

$$\begin{aligned} Y &: 0 \quad\quad 1 \\ P_Y &: \frac{1}{3} \quad\quad \frac{2}{3} \end{aligned}$$

Let X be a Continuous Random Variables with probability density function $f_X(x)$ then what is the pdf of $Y = g(X)$?

$$F_Y(y) = P(Y \leq y) = P\big(g(X) \leq y\big).$$

Can we convert it into known distribution $f_X(x)$?

$$\begin{aligned} F_Y(y) &= P\left(X \leq g^{-1}(y)\right) \quad \text{(is g invertible?)} \\ &= F_X(g^{-1}(y)). \end{aligned}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

Since $F_Y(y) = F_X(g^{-1}(y))$, we have

$$f_Y(y) = f_X\big(g^{-1}(y)\big) \cdot \frac{d}{dy}\big(g^{-1}(y)\big) \quad \text{(is g differentiable?)}$$

Example

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & otherwise. \end{cases}$$

Let $Y = X^2$, find $f_Y(y)$?.

Solution:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P\big(X^2 \leq y\big) \\ &= P\Big(X \leq \sqrt{Y}\Big). \end{aligned}$$

Here $F_Y(y) = F_X(\sqrt{Y})$. Now, we get

$$\begin{aligned} f_Y(y) &= \frac{d}{dY} F_X(\sqrt{Y}) \\ &= f_X(\sqrt{y}).\frac{1}{2\sqrt{Y}} \\ &= \frac{1}{4\sqrt{Y}}, \qquad 0 < y < 4. \end{aligned}$$

Theorem 3. *Let X be a continuous random variable with probability density function $f_X(x)$ and $Y = g(X)$ be a monotonic and differentiable function. Then the probability density function of Y is given by*

$$f_Y(y) = f_X\big(g^{-1}(y)\big)\left|\frac{d}{dy}\,g^{-1}(y)\right|, \quad y \in \mathbb{R}.$$

Proof. Since $g(X)$ is one-to-one and continuous, it is either strictly monotonically increasing or decreasing. Assume that it is strictly monotonic increasing. The cdf of Y is given by

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y)).$$

Hence, the pdf of Y is

$$f_Y(y) = \frac{d}{dy}F_Y(y) = f_X\big(g^{-1}(y)\big)\cdot\frac{d}{dy}\big(g^{-1}(y)\big).$$

In this case, because g is increasing, $\frac{d}{dy}\big(g^{-1}(y) > 0$. Hence we can write $\frac{d}{dy}\big(g^{-1}(y) = |\frac{d}{dy}\big(g^{-1}(y)|$.
Suppose g is strictly decreasing function. Then

$$F_Y(y) = P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y)).$$

Hence, the pdf of Y is

$$f_Y(y) = \frac{d}{dy}F_Y(y) = f_X\big(g^{-1}(y)\big)\cdot -\frac{d}{dy}\big(g^{-1}(y)\big).$$

But since g is decreasing $\frac{d}{dy}\big(g^{-1}(y) < 0$ and, hence $-\frac{d}{dy}\big(g^{-1}(y) = |\frac{d}{dy}\big(g^{-1}(y)|$. Thus, it is true for both cases. $\square$

Example

We consider $f_X(x) = \frac{1}{2}$, $0 < x < 2$ and $Y = e^X$, then find $f_Y(y)$?.

Solution: Since the function $Y = e^X$ is increasing function in given domain so by using theorem 3, we get

$$f_Y(y) = f_X\big(g^{-1}(y)\big)\cdot\frac{d}{dy}\big(g^{-1}(y)\big) = f_X(\ln y)\cdot\frac{1}{y} = \frac{1}{2y}, \quad 1 < y < e^2.$$