Lecture 8

Example

Consider $f_X(x) = 1$, $0 < x < 1$ and $Y = X^3$, then find $f_Y(y)$?.

Solution: Here, Y is a strictly increasing and differentiable function in the given domain. Then from theorem 3, we have

$$f_Y(y) = f_X\big(g^{-1}(y)\big) \cdot \frac{d}{dy}\big(g^{-1}(y)\big) = f_X(y^{1/3}) \cdot \frac{1}{3}y^{-2/3}, = \frac{1}{3y^{2/3}}, \quad 0 < y < 1.$$

Verification: To check whether $f_Y(y) = \dfrac{1}{3}y^{-2/3}$ is a valid pdf or not.
To verify this

1. **Total probability:**

$$\int_0^1 f_Y(y)\, dy = 1.$$

Here

$$\int_0^1 f_Y(y)\, dy = \int_0^1 \frac{1}{3}y^{-2/3}\, dy = \frac{1}{3}\Big[3y^{1/3}\Big]_0^1 = 1.$$

Therefore, $f_Y(y)$ is a valid probability density function.

Note Many times, a function $g(x)$ may not be one-to-one (1–1). For example, consider the function $Y = x^2$. In that case, we see that the region $x \mapsto Y$ is the function's domain and range. So what we do is look at the inverse image for a given Y. And if there are two inverse images, then we split the region, i.e., the domain of x into two disjoint regions, so that both of them map x from each part of the domain to the full range. For that, we have a following result.

Result: Let X be a continuous random variable with pdf $f_X(x)$. Suppose $Y = g(X)$ be piecewise monotonic function of X. If $g_1, g_2, \cdots, g_n$ are piecewise monotonic component of g such that they all map to same range of Y. Then

$$f_Y(y) = \sum_{i=1}^{n} f_X(g_i^{-1}(y))|\frac{d}{dy}(g_i^{-1}(y))|$$

Example

Let $f_X(x) = \frac{1}{2}$, $-1 < x < 1$ and $Y = X^2$.

Solution: $Y = X^2$ is neither increasing nor decreasing in $(-1, 1)$. From $-1 < x < 1$, this maps to $0 \le y < 1$. For a given Y (which is positive), we have two inverse images:

$$x = -\sqrt{y},\ -1 < x < 0 \quad \text{and} \quad x = +\sqrt{y},\ 0 < x < 1.$$

So, if we consider two portions of the domain, both are mapped to same range. So, the idea here is that in each part of the domain, the function will be one-to-one i.e. if we are considering only one inverse image say $+\sqrt{y}$ or $-\sqrt{y}$) then the function is either increasing or decreasing. Therefore, we calculate the density in each region separately and add it. So, we have

$$\begin{aligned} f_Y(y) &= f_X(\sqrt{y})|\frac{d}{dy}\sqrt{y}| + f_X(-\sqrt{y})|\frac{d}{dy}(-\sqrt{y})| \\ &= \frac{1}{2}\frac{1}{2\sqrt{y}} + \frac{1}{2}\frac{1}{2\sqrt{y}} \\ &= \frac{1}{2\sqrt{y}}, \quad 0 < y < 1. \end{aligned}$$

Example

Let $f_X(x) = \dfrac{1}{3}$, $-1 < x < 2$ and $Y = X^2$.

Solution:

From above results, we have

$$f_Y(y) = \begin{cases} f_X(\sqrt{y})\left|\dfrac{d}{dy}\sqrt{y}\right| + f_X(-\sqrt{y})\left|\dfrac{d}{dy}(-\sqrt{y})\right|, & 0 < y < 1, \\ f_X(\sqrt{y})\left|\dfrac{d}{dy}\sqrt{y}\right|, & 1 < y < 4. \end{cases}$$

Since $f_X(x) = \dfrac{1}{3}$ and $\dfrac{d}{dy}\sqrt{y} = \dfrac{1}{2\sqrt{y}}$, we get

$$f_Y(y) = \begin{cases} \dfrac{1}{3\sqrt{y}}, & 0 < y < 1, \\ \dfrac{1}{6\sqrt{y}}, & 1 < y < 4. \end{cases}$$

Example

Let $f_X(x) = \frac{1}{\pi}$, $0 < x < \pi$ and $Y = \sin X$.