Lecture 9

Expectation of a random variable

Definition 6. *Let X be a random variable. If X is a discrete random variable with pmf $p(x)$ and*

$$\sum_x |x|p(x) < \infty,$$

*then the **expectation** of X is*

$$E(X) = \sum_x xp(x).$$

If X is a continuous random variable with pdf $f(x)$ and

$$\int_{-\infty}^{\infty} |x|f(x)dx < \infty,$$

*then the **expectation** of X is*

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

Example

1. Let the random variable X of the discrete type have the pmf given by

$$\begin{array}{lcccc} X: & 0 & 1 & 2 & 3 \\ P_X: & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

   we have

$$E(X) = (0 \times \frac{1}{8}) + (1 \times \frac{3}{8}) + (2 \times \frac{3}{8}) + (3 \times \frac{1}{8}) = \frac{3}{2}.$$

2. Let the random variable X of the continuous type have the pdf given by

$$f(x) = 3x^2, \qquad 0 < x < 1.$$

   Then

$$E(X) = \int_0^1 x \cdot 3x^2 dx = \frac{3}{4}.$$

Definition 7. *Suppose X is a continuous random variable with pdf $f_X(x)$ and let $Y = g(X)$ for some function g. If $\int_{-\infty}^{\infty} |g(x)|f_X(x)dx < \infty$, then the expectation of Y exists and it is given by*

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f_X(x)dx.$$

- **Case:1** If $g(X) = X^r$, then r^{th} moment about the origin is

$$E(X^r) = \sum_{i=1}^{n} x_i^r\, p_i, \qquad \text{(for discrete random variable)}$$

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx, \qquad \text{(for continuous random variable)}$$

- **Case:2** If $g(X) = (X-a)^r$, then r^{th} moment about the point a is

$$E\big[(X-a)^r\big] = \sum_{i=1}^{n} (x_i - a)^r\, p_i, \qquad \text{(for discrete random variable)}$$

$$E(X^r) = \int_{-\infty}^{\infty} (x-a)^r f(x) dx, \qquad \text{(for continuous random variable)}$$

Note: If $r = 1$, then it represents mean.

- **Case:3** Second moment about mean

$$E\big[(X - E(X))^2\big] = \sum_{i=1}^{n} \big[x_i - E(X)\big]^2 p_i.$$

Let $E(X) = \mu$ =mean, then we have

$$E\big[(X - E(X))^2\big] = \sum_{i=1}^{n} \big[x_i - \mu\big]^2 p_i.$$

or

$$E\big[(X - E(X))^2\big] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx.$$

Variance

The variance (measure of randomness) of X is defined to be $E\big[(X - E(X))^2\big]$. It is denoted by σ^2 or by $\text{Var}(X)$.

$$\text{Var}(X) = E\big[(X - E(X))^2\big] = E\big[(X^2 - 2E(X)X + (E(X))^2)\big];$$

since E is a linear operator,

$$\text{Var}(X) = E(X^2) - 2(E(X))^2 + (E(X))^2$$

So,

$$\text{Var}(X) = E(X^2) - \big(E(X)\big)^2.$$

Note: $\text{Var}(X) \geq 0$.

Example

$$X : -1 \quad 0 \quad 1$$
$$P_X : \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}.$$

By solving this $E(X) = 0$ and $E(X^2) = \frac{2}{3}$. Therefore

$$\mathrm{Var}(X) = \sigma^2 = \frac{2}{3}.$$

Standard Deviation

$$\sigma = \sqrt{\mathrm{Var}(X)}.$$

Convex Function

A function $h : \mathbb{R} \to \mathbb{R}$ is said to be **convex** on an interval $I \subseteq \mathbb{R}$ if

$$h\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y),$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Jensen's Inequality

Let $h : \mathbb{R} \to \mathbb{R}$ be a convex function, and let X be a random variable. Then

$$h\big(E(X)\big) \leq E(h(X)).$$

Note: In general,

$$h(E(X)) \neq E(h(X)).$$

Examples:

$$E(e^X) \neq e^{E(X)}, \quad E\left[\frac{1}{X}\right] \neq \frac{1}{E(X)}.$$